Bernoulli Denominators Hidden in Multinomial GCDs

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ABSTRACT

Two candidate-original identities recover Bernoulli-number and Bernoulli-polynomial denominators from a gcd spectrum over positive multinomial coefficients.

Review status. The two Bernoulli bridges proved here are candidate-original consequences of established theorems. The underlying minimum-carry formula is Proposition A.10 of Ando, Hopkins and Strickland; the Bernoulli denominator formulas are also prior. A targeted literature search found nearby multinomial, Stirling-number and digit-sum results, but not the two bridge identities below. That is evidence of novelty, not proof of it. This public research note was developed with substantial LLM assistance and still requires independent review by a number theorist before submission or an unconditional novelty claim.

A single step between two multinomial gcds recovers the denominator of every nonzero even Bernoulli number.

The Bernoulli numbers and multinomial coefficients usually enter mathematics through different doors. Bernoulli numbers control power sums, Euler–Maclaurin summation and special values of the zeta function. Multinomial coefficients count ordered partitions of a finite set. Their definitions do not suggest that the denominator of one should be encoded in the common divisors of the other.

Here is the first indication that they are connected. Take every positive ordered composition of 12 into six parts, form its multinomial coefficient, and take the greatest common divisor. Do the same with seven parts. The two gcds are

$$G_6(12) = 7920, \qquad G_7(12) = 332640.$$

Their ratio is

$$\frac{G_7(12)}{G_6(12)} = 42,$$

which is exactly the denominator of $B_6 = 1/42$. This is not a coincidence.

1 The two bridge identities

For $1 \le r \le n$, define the fixed-part multinomial gcd

$$G_r(n) = \gcd\left\{ \frac{n!}{k_1! \cdots k_r!} : k_i \ge 1, \quad k_1 + \cdots + k_r = n \right\}.$$

Write $\mathrm{den}(q)$ for the positive denominator of a rational number in lowest terms. The first bridge is the exact identity

$$\boxed{\mathrm{den}(B_{2m}) = \frac{G_{2m+1}(4m)}{G_{2m}(4m)}} \qquad (m \ge 1). \tag{1}$$

Thus the denominators

$$6, 30, 42, 30, 66, 2730, \ldots$$

of $B_2, B_4, B_6, B_8, B_{10}, B_{12}, \ldots$ are consecutive ratios along a single diagonal of the multinomial-gcd array.

There is a second bridge for Bernoulli polynomials. Let

$$\mathcal{D}_n = \mathrm{den}\big(B_n(x) - B_n\big)$$

mean the least positive integer that makes every coefficient of the constant-free Bernoulli polynomial integral. Also put

$$\mathcal{P}(n) = \prod_{p \le n} p,$$

the product of all primes at most n. Then

$$\boxed{\mathcal{D}_n \prod_{p \le n} \gcd\big(p, G_p(n)\big) = \mathcal{P}(n),} \tag{2}$$

and the two factors on the left are coprime. Every prime up to n therefore goes to exactly one side: either it appears in the Bernoulli-polynomial denominator, or it divides the multinomial gcd whose number of parts equals that prime, but never both.

Both identities follow from a common object: the complete prime spectrum of $G_r(n)$.

2 The known minimum-carry spectrum

Let $s_p(n)$ be the sum of the base-p digits of n, and let $\nu_p(a)$ be the exponent of the prime p in a. Ando, Hopkins and Strickland proved the minimum-carry formula below as Proposition A.10 in an appendix on additive cocycles (Ando et al., 2001). In the present notation it says

$$\boxed{\nu_p(G_r(n)) = \max\left\{0, \left\lceil \frac{r - s_p(n)}{p-1} \right\rceil\right\}.} \tag{3}$$

Equivalently,

$$G_r(n) = \prod_{p \le n} p^{\max\left\{0, \left\lceil \frac{r-s_p(n)}{p-1} \right\rceil\right\}}. \tag{4}$$

In particular,

$$p \mid G_r(n) \quad \Longleftrightarrow \quad s_p(n) < r.$$

Bergman previously used this zero-versus-positive condition in the language of p-acceptable decompositions (Bergman, 2011). Rowland's multinomial form of Kummer's theorem counts weak compositions at every valuation rather than minimizing over positive ones (Rowland, 2018).

3 A self-contained proof of the spectrum

The bridge proofs only need (3), but the minimum is subtle enough to justify a complete proof. A lower divisibility bound does not determine a gcd unless a coefficient attaining it can also be constructed.

Legendre's factorial formula is

$$\nu_p(a!) = \frac{a - s_p(a)}{p-1}.$$

Applied to a multinomial coefficient, it gives

$$\nu_p!\left(\frac{n!}{k_1! \cdots k_r!}\right) = \frac{\sum_{i=1}^{r} s_p(k_i) - s_p(n)}{p-1}. \tag{5}$$

This is the digit-sum form of the carry principle behind Kummer's theorem (Kummer, 1852). Put

$$S = \sum_{i=1}^{r} s_p(k_i).$$

Every part is positive, so $S \ge r$. Digit sums preserve residue modulo $p-1$, hence

$$S \equiv n \equiv s_p(n) \pmod{p-1}.$$

Equation (5) is nonnegative, so $S \ge s_p(n)$ as well. The smallest number compatible with these restrictions is

$$M = s_p(n) + (p-1)\max\left\{0, \left\lceil \frac{r - s_p(n)}{p-1} \right\rceil\right\}. \tag{6}$$

This proves the lower bound in (3). It remains to attain it.

Write the base-p expansion of n as

$$n = \sum_{j \geq 0} a_j p^j, \qquad 0 \leq a_j \leq p-1.$$

Regard this as a multiset containing a_j copies of p^j. It begins with exactly $s_p(n)$ elements. An element p^j with $j \geq 1$ may be replaced by p copies of p^{j-1}. Each split preserves the sum and increases the number of elements by $p-1$.

The value M in (6) has the required residue and satisfies $M \leq n$: the integer n itself is at least r and is congruent to $s_p(n)$ modulo $p-1$, so it is an eligible upper bound for the least such M. Whenever fewer than n elements remain, at least one is larger than 1 and can be split. Repeated splitting therefore reaches exactly M elements.

Now group the elements into r nonempty blocks. If $r \leq s_p(n)$, group the original base-p elements arbitrarily. At every digit position the whole multiset contains at most $p-1$ elements, so no block creates a carry. If $r > s_p(n)$, the definition of M gives

$$0 \leq M - r \leq p-2.$$

Place $M-r+1$ elements in one block and leave all others as singletons. The non-singleton block contains at most $p-1$ elements, so it also creates no carry. In either case, the resulting positive composition satisfies

$$\sum_{i=1}^{r} s_p(k_i) = M.$$

Substitution in (5) attains the lower bound. The valuation of a gcd is the minimum valuation of its members, so (3) follows for this prime. Repeating for every prime proves (4). □

4 Adjacent gcds form square-free layers

The exponents in (3) never decrease with r. Therefore $G_{r-1}(n) \mid G_r(n)$, and the adjacent layer

$$L_r(n) = \frac{G_r(n)}{G_{r-1}(n)}$$

is an integer. A ceiling can rise by at most one in a single step, so every layer is square-free. More precisely,

$$\boxed{L_r(n) = \prod_{\substack{p \leq n \\ s_p(n) < r \\ p-1 \mid n-r+1}} p.} \tag{7}$$

Indeed, the exponent for p rises at step r exactly when

$$r > s_p(n) \quad \text{and} \quad r - s_p(n) \equiv 1 \pmod{p-1}.$$

Since $s_p(n) \equiv n \pmod{p-1}$, the congruence is the last condition in (7). This known-spectrum corollary is the mechanism behind both bridges.

5 First bridge: the Bernoulli-number rung

For any positive integer d, define the square-free product

$$C_d = \prod_{p-1|d} p.$$

The product is over primes. The key new rung identity is

$$\boxed{L_{d+1}(2d) = C_d.} \tag{8}$$

To prove it, put $n = 2d$ and $r = d + 1$ in (7). A prime in the layer must satisfy $p - 1 \mid d$, so the support of the left side is contained in that of C_d.

For the reverse inclusion, suppose $p - 1 \mid d$ and write $d = t(p - 1)$. Then

$$\nu_p((2d)!) \geq \left\lfloor \frac{2d}{p} \right\rfloor = \left\lfloor \frac{2t(p-1)}{p} \right\rfloor \geq t.$$

Legendre's formula now yields

$$2d - s_p(2d) = (p-1)\nu_p((2d)!) \geq t(p-1) = d.$$

Thus $s_p(2d) \leq d < d + 1$, and p satisfies every condition in (7). Every prime in C_d occurs in the layer. Both sides are square-free, proving (8).

For positive even d, the von Staudt–Clausen theorem states

$$\operatorname{den}(B_d) = \prod_{p-1|d} p = C_d$$

(Clausen, 1840; von Staudt, 1840). Substituting $d = 2m$ into (8) gives the headline identity (1). Notice what the proof does and does not use: it uses only the denominator theorem for Bernoulli numbers, not a recurrence or an approximation to their values.

The first cases are

d	2	4	6	8	10	12
$G_{d+1}(2d)/G_d(2d)$	6	30	42	30	66	2730
$\operatorname{den}(B_d)$	6	30	42	30	66	2730.

6 The whole tail stabilizes

Equation (8) is one point on an infinite constant tail. Fix $d \geq 1$ and let $n \geq d+1$. Reading the layer whose distance from the top is d gives

$$\boxed{L_{n-d+1}(n) = \prod_{\substack{p-1 \mid d \\ p^{d/(p-1)} \mid n!}} p.} \tag{9}$$

To see this, set $r = n-d+1$ in (7). Once $p-1 \mid d$, the remaining digit condition is

$$s_p(n) < n - d + 1.$$

Both $n - s_p(n)$ and d are multiples of $p-1$, so this strict inequality is equivalent to

$$n - s_p(n) \geq d,$$

or, by Legendre's formula,

$$\nu_p(n!) \geq \frac{d}{p-1}.$$

That is precisely the factorial-divisibility test in (9).

If $n \geq 2d$, every prime with $p-1 \mid d$ passes the test by the same floor estimate used above. Consequently

$$L_{n-d+1}(n) = C_d \qquad (n \geq 2d). \tag{10}$$

For even d, every sufficiently high multinomial-gcd ladder therefore has the denominator of B_d on its dth reverse rung.

The exact start of stabilization also has a closed description. If

$$S(a) = \min\{N : a \mid N!\},$$

then the first n from which the reverse rung remains equal to C_d is

$$N_d = \max_{p-1 \mid d} S\left(p^{d/(p-1)}\right). \tag{11}$$

The prime 2 ensures $N_d > d$, so the required layer always exists. The bound $N_d \leq 2d$ follows from the proof of (10). For example, the $d = 6$ tail is $7, 14, 42, 42, \ldots$ at $n = 7, 8, 9, 10, \ldots$: it stabilizes to $\mathrm{den}(B_6) = 42$ already at $N_6 = 9$, before the universal bound $2d = 12$.

7 Second bridge: the Bernoulli-polynomial diagonal

The Bernoulli polynomials are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Kellner and Sondow proved a square-free digit-sum formula for the denominator of the constant-free polynomial (Kellner & Sondow, 2017). Kellner subsequently gave a short p-adic proof and studied the prime product in its own right (Kellner, 2017):

$$\mathcal{D}_n = \operatorname{den}\big(B_n(x) - B_n\big) = \prod_{s_p(n)\geq p} p. \tag{12}$$

The product is finite, and every prime in it is at most n. Meanwhile, the multinomial support criterion at the diagonal value $r = p$ says

$$p \mid G_p(n) \quad \Longleftrightarrow \quad s_p(n) < p. \tag{13}$$

For each prime $p \leq n$, the inequalities in (12) and (13) are exact complements. Therefore

$$\mathcal{D}_n = \prod_{\substack{p\leq n \\ p\nmid G_p(n)}} p, \qquad \prod_{p\leq n} \gcd\big(p, G_p(n)\big) = \prod_{\substack{p\leq n \\ p\mid G_p(n)}} p.$$

The prime supports are disjoint and cover all primes up to n. Multiplying proves (2), including coprimality.

There is a geometric way to read the same result. Prime p first enters the nth gcd ladder at rung

$$r = s_p(n) + 1.$$

If that rung lies at or below the diagonal $r = p$, then p goes to the multinomial factor in (2). If it lies above the diagonal, then p goes to the Bernoulli-polynomial denominator. The diagonal cuts the primorial into two complementary square-free pieces.

At $n = 20$, for example,

$$\mathcal{D}_{20} = 42 = 2 \cdot 3 \cdot 7,$$

while the diagonal multinomial factor contains

$$5 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$$

Their product is

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = \mathcal{P}(20).$$

Bordellès, Luca, Moree and Shparlinski established detailed results about the size and distribution of the prime factors in $\mathcal{D}_n$ (Bordellès et al., 2018). Identity (2) transfers those results into statements about which prime-indexed gcds fail the diagonal divisibility test.

8 The full Bernoulli polynomial is also encoded

The constant term of $B_n(x)$ is B_n, so its denominator is

$$\operatorname{den}(B_n(x)) = \operatorname{lcm}\left(\mathcal{D}_n, \operatorname{den}(B_n)\right).$$

For even $n \geq 2$, the first bridge gives

$$\operatorname{den}(B_n) = \frac{G_{n+1}(2n)}{G_n(2n)},$$

and the second bridge gives

$$\mathcal{D}_n = \frac{\mathcal{P}(n)}{\prod_{p \leq n} \gcd\big(p, G_p(n)\big)}.$$

For odd $n > 1$, $B_n = 0$, so only the second factor is needed. The gcd spectrum therefore determines the denominators of the Bernoulli numbers, their constant-free polynomials, and the complete Bernoulli polynomials.

9 Where the bridge sits in the literature

None of the ingredients should be mistaken for new. Kummer's carry theorem is classical (Kummer, 1852). The exact positive-composition minimum in (3) is Ando, Hopkins and Strickland's Proposition A.10 (Ando et al., 2001). Bergman's criterion already identifies when a prime divides every proper fixed-part multinomial coefficient (Bergman, 2011). The prime products in the Bernoulli-number and Bernoulli-polynomial denominators are respectively the von Staudt–Clausen theorem (Clausen, 1840; von Staudt, 1840) and the Kellner–Sondow formula (Kellner & Sondow, 2017; Kellner, 2017).

There is also a close prior gcd representation of the Bernoulli-number denominator. If $S(N,k)$ is a Stirling number of the second kind, then $k!S(N,k)$ counts surjections from an N-element set onto a k-element set. Komatsu, Luca and Pita Ruiz proved

$$\gcd_{2 \leq k \leq 2m+1} \big(k!S(2m+1,k)\big) = \operatorname{den}(B_{2m})$$

(Komatsu et al., 2014). A surjection count is a sum of positive-part multinomial coefficients. Their theorem takes a gcd across those sums. In contrast, (1) takes two gcds across the individual coefficients and divides the consecutive gcds. The constructions, indices and proofs are different, but the Stirling theorem is the closest

known neighbour and an important warning against presenting every combinatorial representation of the same classical prime product as unprecedented.

That neighbour belongs to an older line of work on tail gcds of the same surjection counts. Lundell studied the prime-power divisibility of

$$\Delta_{N,j} = \gcd\{k!S(N,k) : j \le k \le N\}$$

(Lundell, 1978). Nishimura later proved recurrence and divisibility results relating individual $k!S(N,k)$ to these tail gcds (Nishimura, 2022). Both remain on the Stirling side: they take gcds of sums of positive-part multinomial coefficients, rather than gcds of the individual coefficients with a fixed number of parts.

The candidate-original contribution of this note is limited to the bridge:

1. the exact rung identity (8), and hence the Bernoulli denominator ratio (1);
2. the factorial-filtered tail formula (9), its stabilization (10), and the exact threshold (11);
3. the diagonal complement identity (2); and
4. the resulting reconstruction of full Bernoulli-polynomial denominators from the multinomial-gcd spectrum.

The literature audit searched combinations of "Bernoulli denominator," "multinomial gcd," "minimum carries," "positive compositions," "consecutive gcd ratios," "Stirling gcd," "primorial complement," "digit sum" and "von Staudt–Clausen," then inspected the closest papers, citation graphs and reference chains. Full-text checks found no mention of Bernoulli objects in Bergman's multinomial paper, no multinomial-gcd construction in the Kellner–Sondow denominator papers, and no fixed-part multinomial-gcd bridge in the Lundell–Komatsu–Nishimura Stirling line. This makes the bridge a serious novelty candidate, not a certified first publication.

10 Computational verification

The proofs above are symbolic. Exact arithmetic was nevertheless used to test the indices, strict inequalities and boundary cases independently.

First, the defining multinomial coefficients were enumerated directly for $m = 1, \dots, 5$. For each m, both gcds in (1) were computed without using the spectrum formula. This covered 198,626 positive compositions in total and returned the ratios $6, 30, 42, 30, 66$, matching exact Bernoulli denominators computed from their rational recurrence.

Second, the coefficients of $B_n(x) - B_n$ were constructed as exact rational numbers for every $1 \le n \le 200$. Their least common denominators matched (12), and the two factors in (2) were coprime and multiplied to $\mathcal{P}(n)$ in all 200 cases.

Third, (9) was checked for every even $2 \le d \le 120$ and every $d + 1 \le n \le \min(4d, 200)$: 5,940 tail positions. The factorial filter and the layer support agreed in

every case. Among them, all 2,550 positions with $n \geq 2d$ equalled the corresponding Bernoulli denominator.

Finally, an independent Python/SymPy implementation targeted scale and edge cases rather than direct composition enumeration. It verified (8) for every $1 \leq d \leq 5000$, including the $d = 1$, $p = 2$ and $p = d + 1$ boundaries; matched (1) to the exact denominators of B_2 through B_{400}; verified the exact threshold (11), both before and after stabilization, for every $1 \leq d \leq 500$; and checked the prime-by-prime complement in (2) for every $1 \leq n \leq 5000$. No counterexample was found.

These checks do not prove novelty, and they are not substitutes for the proofs. Their role is narrower: to make an off-by-one error or a failed edge case much less likely to survive into the claim.

11 What has been found

The new object is not a primality test or a method for computing Bernoulli numbers efficiently. Formula (1) replaces a simple classical prime product with gcds of much larger integers, so it is computationally perverse if the only objective is a denominator.

Its value is structural. A theorem about carries in base p, a theorem about the fractional parts of Bernoulli numbers, and a theorem about denominators of power-sum polynomials meet inside one two-dimensional gcd spectrum. Reading a fixed distance from the top produces Bernoulli-number denominators. Cutting the spectrum along the prime diagonal produces the complementary factor of a primorial and hence the Bernoulli-polynomial denominator.

That is a clean bridge between three subjects that were previously adjacent through digit sums but not, in the literature located here, joined by these identities. The algebra is complete. Whether the bridge is genuinely new in the historical record remains the one claim that proof alone cannot settle.

References

Ando, M., Hopkins, M. J., & Strickland, N. P. (2001). Elliptic spectra, the Witten genus and the theorem of the cube. *Inventiones Mathematicae, 146*(3), 595-687. https://doi.org/10.1007/s002220100175

Bergman, G. M. (2011). On common divisors of multinomial coefficients. *Bulletin of the Australian Mathematical Society, 83*(1), 138-157. https://doi.org/10.1017/S0004972710001723

Bordellès, O., Luca, F., Moree, P., & Shparlinski, I. E. (2018). Denominators of Bernoulli polynomials. *Mathematika, 64*(2), 519-541. https://doi.org/10.1112/S0025579318000153

Clausen, T. (1840). Lehrsatz aus einer Abhandlung über die Bernoullischen Zahlen. *Astronomische Nachrichten, 17*(22), 351-352. https://doi.org/10.1002/asna.18400172205

Kellner, B. C. (2017). On a product of certain primes. *Journal of Number Theory, 179*, 126-141. https://doi.org/10.1016/j.jnt.2017.03.020

Kellner, B. C., & Sondow, J. (2017). Power-sum denominators. *The American Mathematical Monthly, 124*(8), 695-709. https://doi.org/10.4169/amer.math.monthly.124.8.695

Komatsu, T., Luca, F., & Pita Ruiz V., C. de J. (2014). A note on the denominators of Bernoulli numbers. *Proceedings of the Japan Academy, Series A, Mathematical Sciences, 90*(5), 71-72. https://doi.org/10.3792/pjaa.90.71

Kummer, E. E. (1852). Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen. *Journal für die reine und angewandte Mathematik, 44*, 93-146. https://doi.org/10.1515/crll.1852.44.93

Lundell, A. T. (1978). A divisibility property for Stirling numbers. *Journal of Number Theory, 10*(1), 35-54. https://doi.org/10.1016/0022-314X(78)90005-7

Nishimura, O. (2022). A formula on Stirling numbers of the second kind and its application to the unstable K-theory of stunted complex projective spaces. *Kyoto Journal of Mathematics, 62*(4). https://doi.org/10.1215/21562261-2022-0026

Rowland, E. (2018). A matrix generalization of a theorem of Fine. *Integers, 18A*, Article A18. https://arxiv.org/abs/1704.05872

von Staudt, K. G. C. (1840). Beweis eines Lehrsatzes, die Bernoullischen Zahlen betreffend. *Journal für die reine und angewandte Mathematik, 21*, 372-374. https://eudml.org/doc/147124